1 Rank of matrices

1.1 Rank and and linearly independent vectors

Recall that we have defined:

Definition 1. Suppose that the system $S = \{v_1, v_2, \dots v_r\}$ is a set of two or more vectors in the vector space $V = \mathbb{R}^n$, then S is said to be a linearly independent set if no vector in S can be expressed as a linear combination of the others. A set that is not linearly independent is said to be linearly dependent.

And we have the characterization:

Theorem 2. *A nonempty set $S = \{v_1, v_2, \dots v_r\}$ in the vector space $V = \mathbb{R}^n$ is linearly independent if and only if the only coefficients satisfying the vector equation*

$$k_1 v_1 + k_2 v_2 + \cdots + k_r v_r = 0$$

are $k_1 = 0, k_2 = 0, \dots, k_r = 0$.

and the property:

Theorem 3. *Let $S = \{v_1, v_2, \dots v_r\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$ then vectors in S are linearly dependent.*

Definition 4. The rank of a matrix A, denoted, $\mathrm{rk}(A)$ is the maximum number of linearly independent columns of A. That is the dimension of the column space.

Remark 5. The rank of a matrix A cannot be higher than the number of columns of A.

Definition 6. A minor of order k of the matrix A is obtained by deleting all but k rows and k columns from A and finding the the determinant of the resulting $k \times k$-matrix.

Example 7. Consider the matrix $\begin{pmatrix} 1 & -2 & 0 & 3 \\ 2 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$. We have 12 minors of order 1 given by each element of A, 18 minors of order 2 when we are left only with two rows and two columns of A like $\begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix} = 2$

We have 4 different minors of order 3 given by:

$$\begin{vmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -7, \quad \begin{vmatrix} -2 & 0 & 3 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 1 & -2 & 3 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}$$

Theorem 8. *The rank of a matrix can be obtained as the order of the highest minor that is different from zero.*

Corollary 9. *The rank of a matrix A cannot be higher than the number of rows of the matrix A.*

Example 10. In our example with the matrix $A = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 2 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$. We know that $r(A) \leq 3$ because A is a 3×4-matrix. Since $\begin{vmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -7$, we get $\mathrm{rk}(A) = 3$.

Example 11. Find the rank of $A = \begin{pmatrix} 1 & -2 & 0 & -3 \\ 2 & -4 & 1 & -6 \\ -3 & 6 & -1 & 9 \end{pmatrix}$. In this example, columns 1,2 and 4 are proportional to column 1. All minors of order 3 will be zero. On the other hand, it is not hard to find a minor order 2 that is not zero, $\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$ and $\mathrm{rk}(A) = 2$.

Example 12. Find the rank of $A_\lambda = \begin{pmatrix} 5-\lambda & 2 & 1 \\ 2 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix}$. We find the determinant of our square matrix, to determine for which values of λ is that determinant different from zero:

$$\begin{vmatrix} 5-\lambda & 2 & 1 \\ 2 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = \lambda(1-\lambda)(\lambda-6).$$

Hence for $\lambda \neq 0, 1, 6$, the rank of our matrix A_λ is $\mathrm{rk}(A_\lambda) = 3$. We can check that for the values $0, 1$ and 6, the rank is 2.

Theorem 13. *The rank of a matrix is the same as the rank of its transpose. The rank of a matrix can be obtained as the maximum numbers of linearly independent rows.*

Remark 14. The rank of a matrix is not affected by row operations. The most efficient way to find rank of a matrix is to perform row-operations to reduce the matrix.

Theorem 15. *A linear system of equations is consistent if and only if the rank of the matrix coincide with the rank of the augmented matrix.*

1.2 Dimension theorem for matrices

Definition 16. The null space or kernel of a matrix $A_{m\times n}$ is the subspace

$$\ker(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

of solutions to the homogeneous system of equations defined by A. The dimension of the null space of A is called the nullity of A.

Theorem 17. *(Dimension Theorem for Matrices) If A is a matrix with n columns, then*

$$rk(A) + nullity(A) = n.$$

Proof. Since A has n columns, the homogeneous linear system $Ax = 0$ has n unknowns (variables). These fall into two distinct categories: the leading variables and the free variables. Thus,

$$\text{numer of leading variables } + \text{ number of free variables } = n.$$

But the number of leading variables is the same as the number of leading 1's in any row echelon form of A, which is the same as the dimension of the row space of A, which is the same as the rank of A. Also, the number of free variables in the general solution of $Ax = 0$ is the same as the number of parameters in that solution, which is the same as the dimension of the solution space of $Ax = 0$, which is the same as the nullity of A. □